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Complex Zolotarev Polynomials on the Real Interval [−1, 1]

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We consider complex Zolotarev polynomials of degree n on $[-1, 1]$, i.e., monic polynomials of degree n with the second coefficient assigned to a given complex number ρ, that have minimum Chebyshev norm on $[-1, 1]$. They can be characterized either by n or by $n+1$ extremal points. We show that those corresponding to n extrema are closely related to real Zolotarev polynomials on $[-1, 1]$, so that we distinguish between a trigonometric case where an explicit expression is given and the more complicated elliptic case. The classification of the parameters ρ that lead to one of the above cases is provided.

1. INTRODUCTION AND NOTATIONS

Given an integer $n \geqslant 2$, $n \in \mathbb{N}$, and a complex number $\rho = \sigma + i\tau$, $(\sigma, \tau) \in \mathbb{R}^2$, the Zolotarev polynomial $Z_n(z, \rho)$ on $[-1, 1]$ is the complex polynomial of degree n whose first two coefficients are equal to 1 and ρ, that deviates least from zero on $[-1, 1]$. More precisely, its Chebyshev norm $\|Z_n(\rho)\|$ on $[-1, 1]$ satisfies

$$\|Z_n(\rho)\| = \min \Big\{ \|p_n\|, p_n(z) = \sum_{j=0}^{n} a_j z^j, a_n = 1, \\ a_{n-1} = \rho, (a_{n-2}, ..., a_0) \in \mathbb{C}^{n-1} \Big\},$$

with $\|p_n\| = \max\{|p_n(z)|, z \in [-1, 1]\}$.

By symmetry, it suffices to consider $\sigma \geqslant 0$ and $\tau \geqslant 0$. Indeed, from $Z_n(z, \rho)$, a simple computation yields $Z_n(z, -\rho) = (-1)^n Z_n(-z, \rho)$, $Z_n(z, \bar{\rho}) = \overline{Z_n(z, \rho)}$ and, consequently, $Z_n(z, -\bar{\rho}) = (-1)^n \overline{Z_n(-z, \rho)}$, where the upper bar stands for complex conjugacy.

The problem originally stated and solved by Zolotarev refers to $\rho \in \mathbb{R}$, i.e., $\rho = \sigma$ [1-3, 6]. As is discussed, for instance, by Carlson and Todd in

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their expository paper [3], there is a critical value $\gamma = n\tan^2[\pi/(2n)]$ such that the solution is expressed in terms of trigonometric functions for $\sigma \leqslant \gamma$ and in terms of elliptic functions for $\sigma \geqslant \gamma$.

Recently, for purely imaginary values of ρ, i.e., $\rho = i\tau$, an explicit expression of $Z_n(z, i\tau)$ has been obtained by Freund [5] when $\tau \leqslant 1$ and by the authors [9] when $\tau \geqslant 1$. If $T_k(z)$, $k \in \mathbb{N}$, denotes the Chebyshev polynomial of the first kind, i.e., $T_k(z) = \cos k\vartheta$, $z = \cos\vartheta$, then

$$Z_n(z, i\tau) = 2^{1-n}[T_n(z) + 2i\tau T_{n-1}(z) - \delta T_{n-2}(z)], \tag{1}$$

where $\delta = \tau^2$ for $\tau \leqslant 1$ and $\delta = 1$ for $\tau \geqslant 1$.

The object of the present work is to investigate $Z_n(z, \rho)$, $\rho = \sigma + i\tau$, for nonzero values of σ and τ.

By the well-established theory of uniform approximation by complex polynomials, $Z_n(z, \rho)$ exists, is unique, and satisfies the following characterization [7, 8].

THEOREM 1. *The Zolotarev polynomial $Z_n(z, \rho)$ on $[-1, 1]$ is characterized by m extremal points $z_j \in [-1, 1]$, $j = 1, 2, ..., m$, with $n \leqslant m \leqslant 2n-1$, such that*

$$Z_n(z_j, \rho) = \varepsilon_j \, \|Z_n(\rho)\|, \qquad |\varepsilon_j| = 1, \qquad j = 1, 2, ..., m,$$

with

$$\sum_{j=1}^{m} s_j p_{n-2}(z_j) = 0, \qquad s_j \neq 0, \quad \operatorname{sgn} s_j = \overline{\varepsilon_j}, \quad \textit{all } p_{n-2}, \tag{2}$$

where $\operatorname{sgn} s_j = s_j/|s_j|$.

For the problem at hand, it is not hard to verify that $Z_n(z, \rho)$ has at most $n+1$ extrema in $[-1, 1]$. Hence, the parameters ρ can be classified in a set A for which $m = n$ and in a set B for which $m = n+1$. For example, $\rho = \sigma \in A$ for all $\sigma \geqslant 0$ whereas $\rho = i\tau$ is in A for $\tau \geqslant 1$ and in B for $\tau \in (0, 1)$. In Section 2, we show that, for any $\rho \in A$, $Z_n(z, \rho)$ is connected with real Zolotarev polynomials. This leads us to distinguish between a trigonometric case for which an explicit expression of $Z_n(z, \rho)$ is given in Section 3 and the more complicated elliptic case that is treated in Section 4.

2. RELATIONSHIP BETWEEN REAL AND COMPLEX ZOLOTAREV POLYNOMIALS FOR $\rho \in A$

When $m = n$, the coefficients s_j in (2) are simply given by $c/Q'(z_j)$ with $c \neq 0$ and $Q(z) = \prod_{k=1}^{n} (z - z_k)$ [7, 8]. This yields $\varepsilon_j = \eta \operatorname{sgn} Q'(z_j)$, where

$\eta = \operatorname{sgn} \bar{c}$. Moreover, there holds an expression of $Z_n(z, \rho)$ in terms of its n extremal points.

THEOREM 2. *For $\rho \in A$, $Z_n(z, \rho)$ is given by*

$$Z_n(z, \rho) = \eta \, \|Z_n(\rho)\| \sum_{j=1}^{n} |Q'(z_j)|^{-1} \, Q(z)/(z - z_j) + Q(z), \tag{3}$$

with

$$\eta \, \|Z_n(\rho)\| = \left(\rho + \sum_{j=1}^{n} z_j\right) \Big/ \sum_{j=1}^{n} |Q'(z_j)|^{-1}. \tag{4}$$

Proof. By (3), $Z_n(z, \rho)$ is a monic polynomial of degree n and, by (4), its second coefficient is ρ. From (3), we easily find $Z_n(z_j, \rho) = \eta \operatorname{sgn} Q'(z_j) \, \|Z_n(\rho)\|$, as required. ∎

In the sequel, we order the extremal points with increasing values, i.e., $z_1 < z_2 < \cdots < z_n$, so that $\operatorname{sgn} Q'(z_j) = (-1)^{n-j}$, $j = 1, 2, ..., n$. To determine them, we need the following result that is basic for the remainder of the paper.

THEOREM 3. *For $\rho = \sigma + i\tau \in A$, the extremal points $z_1, z_2, ..., z_n$ of $Z_n(z, \rho)$ are those of the real Zolotarev polynomial $Z_n(z, r)$ where the real parameter r is related to σ and τ by*

$$r = \sigma + \tau^2 \Big/ \left(\sigma + \sum_{j=1}^{n} z_j\right). \tag{5}$$

Proof. Setting $\eta = e^{i\vartheta}$, we find

$$Z_n(z_j, \rho) = e^{i\vartheta}(-1)^{n-j} \, \|Z_n(\rho)\|, \qquad j = 1, 2, ..., n,$$

or

$$q_n(z_j) = (-1)^{n-j} \, \|Z_n(\rho)\|, \qquad j = 1, 2, ..., n, \tag{6}$$

where $q_n(z_j) = \mathscr{Re}\{e^{-i\vartheta} Z_n(z_j, \rho)\}$. For $z \in \mathbb{R}$, $q_n(z)$ is a real polynomial of degree n whose first two coefficients are $\cos \vartheta$ and $\sigma \cos \vartheta + \tau \sin \vartheta$. Furthermore, for $z \in [-1, 1]$, we have $|q_n(z)| \leqslant |Z_n(z, \rho)| \leqslant \|Z_n(\rho)\|$ so that, by (6), $q_n(z)$ assumes its maximum value with alternating signs at $z_1, z_2, ..., z_n$. By virtue of the equioscillation theorem [2, 6], $q_n(z) = \cos \vartheta \, Z_n(z, r)$, where $r = \sigma + \tau \tan \vartheta$. In view of (4), we compute $\tan \vartheta = \tau/(\sigma + \sum_{j=1}^{n} z_j)$ and we obtain (5). ∎

Note that, for given r, Eq. (5) defines a circle C_r in the ρ-plane. For

$\tau=0$, r is evidently equal to σ. For $\tau\neq 0$ and $\sigma\to 0$, $\sum_{j=1}^{n} z_j\to 0$ by symmetry, so that $r\to\infty$ and $z_1, z_2, ..., z_n$ tend to the extremal points of $T_{n-1}(z)$ as was proved in [9] for (1) when $\tau\geqslant 1$ and $\delta=1$.

The above two theorems will serve for determining all Zolotarev polynomials associated with $\rho\in A$. Given $r>0$, we use the extremal points $z_1, z_2, ..., z_n$ of $Z_n(z, r)$ to consider

$$p_n(z)=P\sum_{j=1}^{n}|Q'(z_j)|^{-1}Q(z)/(z-z_j)+Q(z), \tag{7}$$

with

$$P=\left(\rho+\sum_{j=1}^{n}z_j\right)\Big/\sum_{j=1}^{n}|Q'(z_j)|^{-1}. \tag{8}$$

For all values of ρ on C_r such that $\|p_n\|=|P|$, we obtain $p_n(z)=Z_n(z,\rho)$.

In order to carry out this analysis, we first establish a technical lemma.

LEMMA 1. *With the above notations, the following holds for* $\rho\in C_r$

$$|p_n(z)|^2-|P|^2=\left(r+\sum_{j=1}^{n}z_j\right)^{-1}\times\left\{\left(\sigma+\sum_{j=1}^{n}z_j\right)[Z_n^2(z,r)-\|Z_n(r)\|^2]+(r-\sigma)\,Q^2(z)\right\}. \tag{9}$$

Proof. In view of (7) and (8), we have

$$|p_n(z)|^2=\left[\left(\sigma+\sum_{j=1}^{n}z_j\right)^2+\tau^2\right]R^2(z)+2\left(\sigma+\sum_{j=1}^{n}z_j\right)R(z)\,Q(z)+Q^2(z), \tag{10}$$

where

$$R(z)=\left[\sum_{j=1}^{n}|Q'(z_j)|^{-1}\right]^{-1}\sum_{j=1}^{n}|Q'(z_j)|^{-1}Q(z)/(z-z_j). \tag{11}$$

Now, $Z_n(z,r)$ is obtained by putting $\rho=r$ in (3) and (4), so that

$$Z_n^2(z,r)=\left(r+\sum_{j=1}^{n}z_j\right)^2R^2(z)+2\left(r+\sum_{j=1}^{n}z_j\right)R(z)\,Q(z)+Q^2(z).$$

Hence, making use of (5), we write (10) in the form

$$|p_n(z)|^2 = \left(r + \sum_{j=1}^{n} z_j\right)^{-1} \left[\left(\sigma + \sum_{j=1}^{n} z_j\right) Z_n^2(z, r) + (r-\sigma)\, Q^2(z)\right]. \quad (12)$$

Combining (4) in which $\rho = r$, (5) and (8) yields

$$|P|^2 = \left(r + \sum_{j=1}^{n} z_j\right)^{-1} \left(\sigma + \sum_{j=1}^{n} z_j\right) \|Z_n(r)\|^2,$$

and, by subtraction with (12), the quoted result (9). ∎

We shall partition A in a set A_1 when $r \leqslant \gamma$ and a set A_2 when $r > \gamma$, thereby defining the trigonometric and elliptic cases treated in the next two sections.

3. The Trigonometric Case

For $r \leqslant \gamma$, the explicit expression of $Z_n(z, r)$ is [3, Theorem 1]

$$Z_n(z, r) = 2^{1-n}(1 + rn^{-1})^n\, T_n(x), \qquad x = (z + rn^{-1})/(1 + rn^{-1}). \quad (13)$$

Whence we prove

Theorem 4. *For* $0 < r \leqslant \gamma$ *and* $\rho = \sigma + i\tau \in C_r$, $\sigma \geqslant 0$, $\tau \geqslant 0$, $p_n(z)$ *defined in* (7) *and* (8) *is the Zolotarev polynomial* $Z_n(z, \rho)$ *and, consequently,* $\rho \in A_1$, *iff*

$$\sigma \geqslant (n-1)\tau^2. \quad (14)$$

Proof. By virtue of (13), $\|Z_n(r)\| = 2^{1-n}(1 + rn^{-1})^n$ and $z_j = (1 + rn^{-1}) \cos[(n-j)\,\pi/n] - rn^{-1}$, $j = 1, 2, ..., n$, so that

$$\sum_{j=1}^{n} z_j = 1 + r(n^{-1} - 1). \quad (15)$$

On substituting these values in (9), we obtain after some computation

$$|p_n(z)|^2 - |P|^2 = \prod_{j=1}^{n-1} (z - z_j)^2\, (z-1)(z-\omega),$$

where $\omega = -1 - 2rn^{-1} + 2(r - \sigma)$. Clearly, we have $|p_n(z)| \leqslant |P|$ for all $z \in [-1, 1]$ iff $\omega \leqslant -1$ or, equivalently, iff the following inequality holds

$$\sigma \geqslant (1 - n^{-1})r. \quad (16)$$

To show that (14) and (16) are equivalent, we insert (15) in (5) to obtain the equation of C_r

$$\tau^2 - (r-\sigma)(\sigma - r + 1 + rn^{-1}) = 0. \tag{17}$$

Rewriting (17) as $\tau^2 = rn^{-1} - [\sigma - (1-n^{-1})r][1 - r + \sigma]$, and assuming (16) which implies $\sigma \geqslant r-1$ since $n^{-1}r \leqslant \tan^2[\pi/(2n)] \leqslant 1$, we conclude that $\tau^2 \leqslant rn^{-1} \leqslant (n-1)^{-1}\sigma$ as required. Conversely, starting from $\sigma \geqslant (n-1)\tau^2$ where τ^2 is given by (17), we obtain $[\sigma - (1-n^{-1})r]$ $[n(n-1)^{-1} - r + \sigma] \geqslant 0$ or $\sigma \geqslant (1-n^{-1})r$ because $n(n-1)^{-1} - r + \sigma \geqslant n\{(n-1)^{-1} - \tan^2[\pi/(2n)]\} \geqslant 0$. ∎

We conclude by stating

THEOREM 5. *Let $\rho = \sigma + i\tau$, $\sigma \geqslant 0$, $\tau \geqslant 0$, such that $\tau^2 \leqslant (\gamma - \sigma)(\sigma - \gamma + 1 + \gamma n^{-1})$, $\gamma = n\tan^2[\pi/(2n)]$. If $\tau^2 > \sigma(n-1)^{-1}$, then $\rho \in B$. For $\tau^2 \leqslant \sigma(n-1)^{-1}$, $\rho \in A_1$, and $Z_n(z, \rho)$ is explicitly given by*

$$\begin{aligned} Z_n(z,\rho) = {} & 2^{1-n}(1 + r^*n^{-1})^{n-1}\{(1 + r^*n^{-1})\,T_n(x) \\ & + (\rho - r^*)[U_{n-1}(x) - U_{n-2}(x)]\}, \end{aligned} \tag{18}$$

where $x = (1 + r^*n^{-1})^{-1}(z + r^*n^{-1})$ and, for $k \in \mathbb{N}$, $U_k(x)$ denotes the Chebyshev polynomial of the second kind, i.e., $U_k(x) = \sin[(k+1)\vartheta]/\sin\vartheta$, $x = \cos\vartheta$. The real constant r^* in (18) is

$$r^* = \sigma + [2(n-1)]^{-1}(\sigma + n - nv^{1/2}) \tag{19}$$

with $v = (n^{-1}\sigma + 1)^2 - 4(1 - n^{-1})\tau^2$.

Proof. Given σ and τ, the left-hand side of (17) is a quadratic polynomial in r, denoted by $g(r)$. As $g(0) = \tau^2 + \sigma^2 + \sigma \geqslant 0$ and $g(\gamma) \leqslant 0$ by hypothesis, $g(r)$ vanishes at some $r^* \in [0, \gamma]$ so that $\rho \in C_{r^*}$. By virtue of Theorem 4, ρ is in B for $\tau^2 > \sigma(n-1)^{-1}$ and in A_1 for $\tau^2 \leqslant \sigma(n-1)^{-1}$. If $g(\gamma) < 0$, as $g(r) \to +\infty$ for $r \to +\infty$, r^* is the smallest root of $g(r)$ given by (19). If $g(\gamma) = 0$, i.e., $r^* = \gamma$, the second root is $-\gamma + (n-1)^{-1}n[1 + (2 - n^{-1})\sigma]$ which is greater than γ for $\rho \in A_1$, i.e., for σ satisfying (16). Hence r^* is also given by (19).

For $\rho \in A_1$, $Z_n(z, \rho)$ is

$$Z_n(z,\rho) = \left(\rho + \sum_{j=1}^{n} z_j\right) R(z) + Q(z), \tag{20}$$

where $Q(z) = \prod_{k=1}^{n}(z - z_k)$, $R(z)$ is defined in (11) and $z_1, z_2, \ldots, z_n$ are the extremal points of $Z_n(z, r^*) = 2^{1-n}(1+r^*n^{-1})^n\, T_n(x)$, $x = (1+r^*n^{-1})^{-1}(z + r^*n^{-1})$. Now, $Z_n(z, r^*)$ is also given by

$$Z_n(z, r^*) = \left(r^* + \sum_{n=1}^{n} z_j\right) R(z) + Q(z). \tag{21}$$

An easy calculation yields

$$Q(z) = 2^{1-n}(1+r^*n^{-1})^n\, n^{-1} T_n'(x)(x-1) \tag{22}$$

and, by (21),

$$R(z) = 2^{1-n}(1+r^*n^{-1})^{n-1}\,[T_n(x) - n^{-1}(x-1)\, T_n'(x)]. \tag{23}$$

On substituting (22) and (23) in (20) and performing some arrangements based on the trigonometric definition of Chebyshev polynomials, we obtain (18). ∎

4. The Elliptic Case

We shall describe $Z_n(z, r)$, $r > \gamma$, in terms of elliptic functions with the notations of Carlson and Todd [3], which are based on those used in the book of Whittaker and Watson [10]. When $r > \gamma$, $Z_n(z, r)$ is given by [1, Theorem 2]

$$Z_n(z, r) = \|Z_n(r)\|\; T_n[(X + X^{-1})/2],$$

where

$$X = -\vartheta_1[(\pi u/2K) - (\pi/2n)]/\vartheta_1[(\pi u/2K) + (\pi/2n)]$$

and

$$\|Z_n(r)\| = 2^{1-n}\{\vartheta_2\vartheta_3/[\vartheta_2(\pi/2n)\,\vartheta_3(\pi/2n)]\}^{2n}$$

such that z is related to u by

$$z = -[\mathrm{sn}^2 u + \mathrm{sn}^2(K/n)]/[\mathrm{sn}^2 u - \mathrm{sn}^2(K/n)].$$

The modulus k of the elliptic functions is the unique solution in $(0, 1)$ of

$$r = n\{2\,\mathrm{sn}(K/n)[\mathrm{cn}(K/n)\,\mathrm{dn}(K/n)]^{-1}\{[\mathrm{sn}(2K/n)]^{-1} - \mathrm{zn}(K/n)\} - 1\}. \tag{24}$$

In addition to the extremal points $z_1 = -1 < z_2 < \cdots < z_n = 1$, there are two points $\alpha < \beta < -1$ at which $|Z_n(z, r)|$ takes on the value $\|Z_n(r)\|$. They are given by

$$\alpha = [\mathrm{sn}^2(K/n) + 1]/[\mathrm{sn}^2(K/n) - 1], \tag{25}$$

$$\beta = [k^2\,\mathrm{sn}^2(K/n) + 1]/[k^2\,\mathrm{sn}^2(K/n) - 1], \tag{26}$$

and they satisfy the relation

$$2^{-1}(\alpha + \beta) + \sum_{j=1}^{n} z_j + r = 0. \tag{27}$$

Now we prove

THEOREM 6. *For $r > \gamma$ and $\rho = \sigma + i\tau \in C_r$, $\sigma \geqslant 0$, $\tau \geqslant 0$, $p_n(z)$ defined in* (7) *and* (8) *is the Zolotarev polynomial $Z_n(z, \rho)$ and, consequently, $\rho \in A_2$, iff $t = \sigma - r \geqslant t_a$, where*

$$t_a = -\left[\frac{(1+k)\,\mathrm{sn}(K/n)}{\mathrm{cn}(K/n)\,\mathrm{dn}(K/n)}\right]^2 \frac{1 - k\,\mathrm{sn}^2(K/n)}{1 + k\,\mathrm{sn}^2(K/n)}. \tag{28}$$

Proof. From the above description of $Z_n(z, r)$, we have

$$Z_n^2(z, r) - \|Z_n(r)\|^2 = (z^2 - 1) \prod_{j=2}^{n-1} (z - z_j)^2 (z - \alpha)(z - \beta), \qquad \alpha < \beta < -1,$$

so that, after some manipulations making use of (27), Eq. (9) becomes

$$|p_n(z)|^2 - |P|^2 = (z^2 - 1) \prod_{j=2}^{n-1} (z - z_j)^2\, h(z),$$

where $h(z)$ is the quadratic polynomial $(z - \alpha)(z - \beta) + 2t[z - (1 + \alpha\beta)/(\alpha + \beta)]$. By Section 2, $p_n(z) = Z_n(z, \rho)$ iff $|p_n(z)| \leqslant |P|$ or, equivalently, $h(z) \geqslant 0$ for all $z \in [-1, 1]$. For $\rho \in C_r$, we have to consider negative values of t. The discriminant of $h(z)$ is $4(t - t_a)(t - t_b)$, $t_a < t_b < 0$, where

$$t_a = [2(\alpha + \beta)]^{-1}\, [(\alpha^2 - 1)^{1/2} + (\beta^2 - 1)^{1/2}]^2,$$
$$t_b = [2(\alpha + \beta)]^{-1}\, [(\alpha^2 - 1)^{1/2} - (\beta^2 - 1)^{1/2}]^2.$$

For $t_b \leqslant t \leqslant 0$, it is easily verified that the roots of $h(z)$ lie in $[\alpha, \beta]$ while, for $t_a < t < t_b$, they are complex. Therefore, $h(z)$ is nonnegative in $[-1, 1]$ for $t_a < t \leqslant 0$. When $t = t_a$, $h(z)$ has a double root at $\lambda = (\alpha + \beta)/2 - t_a$ which, in view of (25) and (26), can be written as $\lambda = -[1 - k\,\mathrm{sn}^2(K/n)]/[1 + k\,\mathrm{sn}^2(K/n)]$. Thus λ is in $(-1, 0)$ for $0 < k < 1$. As $h(\lambda) < 0$ for $t < t_a$,

$h(z)$ is nonnegative in $[-1, 1]$ iff $t \geqslant t_a$. By (25) and (26), t_a is still equal to (28), as asserted. ∎

By virtue of (27), Eq. (5) of C_r can be put in the form

$$\tau^2 - (r-\sigma)[\sigma - r - (\alpha+\beta)/2] = 0. \tag{29}$$

Introducing $\sigma_a(k) = r + t_a$, where r and t_a are defined by (24) and (28), and inserting $\sigma = \sigma_a(k)$ in (29) with α and β given by (25) and (26), yield the positive value of $\tau = \tau_a(k)$ on C_r,

$$\tau_a(k) = \frac{(1+k)\,\mathrm{sn}(K/n)}{\mathrm{cn}(K/n)\,\mathrm{dn}(K/n)}\,\frac{1-k\,\mathrm{sn}^2(K/n)}{1+k\,\mathrm{sn}^2(K/n)}. \tag{30}$$

The set $\{(\sigma_a(k), \tau_a(k)\}, 0<k<1\}$ defines the boundary curve separating domains of the ρ-plane associated with $\rho \in A_2$ and $\rho \in B$. We list several properties of this curve in

PROPERTY 1. *The functions* $\sigma_a(k)$ *and* $\tau_a(k)$ *satisfy*

(a) $(\sigma_a(k), \tau_a(k)) \to ((n-1)\tan^2(\pi/2n), \tan(\pi/2n))$ *as* $k \to 0$.

(b) $(\sigma_a(k), \tau_a(k)) \to (0, 1)$ *as* $k \to 1$.

(c) *for* $n=2$, $(\sigma_a(k), \tau_a(k)) = ([(1-k)/(1+k)]^{1/2}, 1)$, $0<k<1$.

(d) *for* $n>2$, $\sigma_a(k)>0$, *and* $\tau_a(k)<1$, $0<k<1$.

Proof. Properties (a) and (b) are direct consequences of the degenerating behavior of elliptic functions [4]. We prove Property (c) by putting [3, Lemma 2] $\mathrm{sn}(K/2) = (1+k')^{-1/2}$, $\mathrm{cn}(K/2) = [k'/(1+k')]^{1/2}$, $\mathrm{dn}(K/2) = (k')^{1/2}$, $\mathrm{zn}(K/2) = (1-k')/2$, $k' = (1-k^2)^{1/2}$, in (24), (28), and (30). In Property (d), we first show that, for $n>2$, $\tau_a(k)<1$, $0<k<1$. Indeed, squaring the expression on the right of (30) and introducing the variable $y = \mathrm{sn}^2(K/n) \in (0, 1)$, $0<k<1$, yield the function $F(y)$ whose derivative can be put in the form

$$F'(y) = \frac{(1+k)^2\,(1-ky)[(1-ky)^4 + 4k(1-k)^2 y^2]}{(1+ky)^3\,(1-y)^2\,(1-k^2y)^2}$$

Clearly, $F'(y)$ is positive in (0, 1) for $0<k<1$. Since $y = \mathrm{sn}^2(K/n)$, $n>2$, is smaller than $\mathrm{sn}^2(K/2)$ for $0<k<1$, it follows that $\tau_a^2(k) = F(\mathrm{sn}^2(K/n)) < F(\mathrm{sn}^2(K/2)) = 1$, $0<k<1$, as announced. Finally, if Property (d) is not true for $\sigma_a(k)$, there is some $k_0 \in (0, 1)$ such that $\sigma_a(k_0) = 0$ together with $0<\tau_a(k_0)<1$ and $\rho_a = i\tau_a(k_0) \in A_2$. But, it was mentioned in the introduction that $\rho = i\tau$ does not belong to A_2 for $\tau \in (0, 1)$. We easily check this assertion by noting that, by (3), $Z_n(1, \rho)/Z_n(-1, \rho) = (-1)^{n-1}$ if $\rho \in A_2$, whereas $Z_n(z, i\tau)$, $0<\tau<1$, given by (1) in which $\delta=\tau^2$ satisfies $Z_n(1, i\tau)/Z_n(-1, i\tau)=(-1)^n(1-\tau^2+2i\tau)/(1-\tau^2-2i\tau)$. ▮

Summing up, we conclude

COROLLARY 1. *Given $\rho=\sigma+i\tau$, $\sigma>0$, $\tau\geqslant 0$, with $\tau^2>(\gamma-\sigma)(\sigma-\gamma+1+\gamma n^{-1})$, $\gamma=n\tan^2(\pi/2n)$, there exists $k^*\in(0,1)$ such that $\rho\in C_{r^*}$, where r^* is the corresponding value of r defined by* (24). *For $\tau<1$, if $\sigma<\sigma_a(k^*)$, then $\rho\in B$. Otherwise, $\rho\in A_2$ and $Z_n(z,\rho)$ is obtained by introducing the extremal points $z_1, z_2, ..., z_n$ of $Z_n(z, r^*)$ in* (3) *and* (4).

Proof. The result is immediate if we show the existence of $k^*\in(0,1)$. To this end, we denote by $G(k)$, the left-hand side of (29) where r, α, and β are given by (24), (25), and (26), respectively. First, we have $G(0)=\tau^2-(\gamma-\sigma)(\sigma-\gamma+1+\gamma n^{-1})>0$ by hypothesis. Then, as $k\to 1$, $r\to+\infty$ and $z_1, z_2, ..., z_n$ tend to the extremal points of $T_{n-1}(z)$. Thus, using (27), we find $r+(\alpha+\beta)/2=-\sum_{j=1}^{n} z_j\to 0$ when $k\to 1$ so that $G(k)\sim-\sigma r<0$. By continuity, $G(k)$ vanishes at some k^* lying in the open interval (0, 1), as required. ▮

Unfortunately, the explicit values of all extremal points of the real Zolotarev polynomial are not known in the elliptic case (see [3, bottom of p. 25]), except for $n=2$ where z_1 and z_2 are simply -1 and 1. When $n=2$, it is even possible to determine the explicit expression of $Z_n(z,\rho)$ for $\rho\in B$, as is shown in the last theorem.

THEOREM 7. *Let $Z_2(z,\rho)=z^2+\rho z-d$, $\rho=\sigma+i\tau$, $\sigma\geqslant 0$, $\tau\geqslant 0$.*

(a) *If $\tau^2\leqslant\min\{\sigma, 2\sigma-\sigma^2\}$, then $\rho\in A_1$,*

$$d=[4+(r^*)^2+2(2-r^*)\rho]/8$$

with $(z_1, z_2)=(-2^{-1}r^*, 1)$ and

$$\|Z_2(\rho)\|=\frac{2^{-1}(1+2^{-1}r^*)^2}{\{1+[\tau/(\sigma+1-2^{-1}r^*)]^2\}^{1/2}},$$

where

$$r^*=2^{-1}\{(2+3\sigma)-[(\sigma+2)^2-8\tau^2]^{1/2}\}.$$

(b) *If either $\tau^2\geqslant 1$, $(\sigma,\tau)\neq(1,1)$ or $2\sigma-\sigma^2<\tau^2<1$, then $\rho\in A_2$, $d=1$, $(z_1, z_2)=(-1, 1)$, and $\|Z_2(\rho)\|=|\rho|$.*

(c) *If* $\sigma < \tau^2 < 1$, *then* $\rho \in B$,

$$d = 2^{-1}[(1+\tau^2) - i\sigma(\tau - \tau^{-1})],$$

with $(z_1, z_2, z_3) = (-1, -\sigma, 1)$ *and* $\|Z_2(\rho)\| = 2^{-1}(\tau + \tau^{-1})\,|\rho|$.

Proof. Statements (a) and (b) are particular applications of Theorem 5 and Corollary 1. We solve the third case characterized by three extremal points $z_1 = -1 < z_2 < z_3 = 1$, by identifying $|Z_2(z, \rho)|^2 - |Z_2(1, \rho)|^2$ with $(z^2 - 1)(z - z_2)^2$ to obtain the asserted values of d and z_2. ∎

Finally, the three domains A_1, A_2, and B of the ρ-plane are exhibited in Fig. 1 for several values of the degree n.

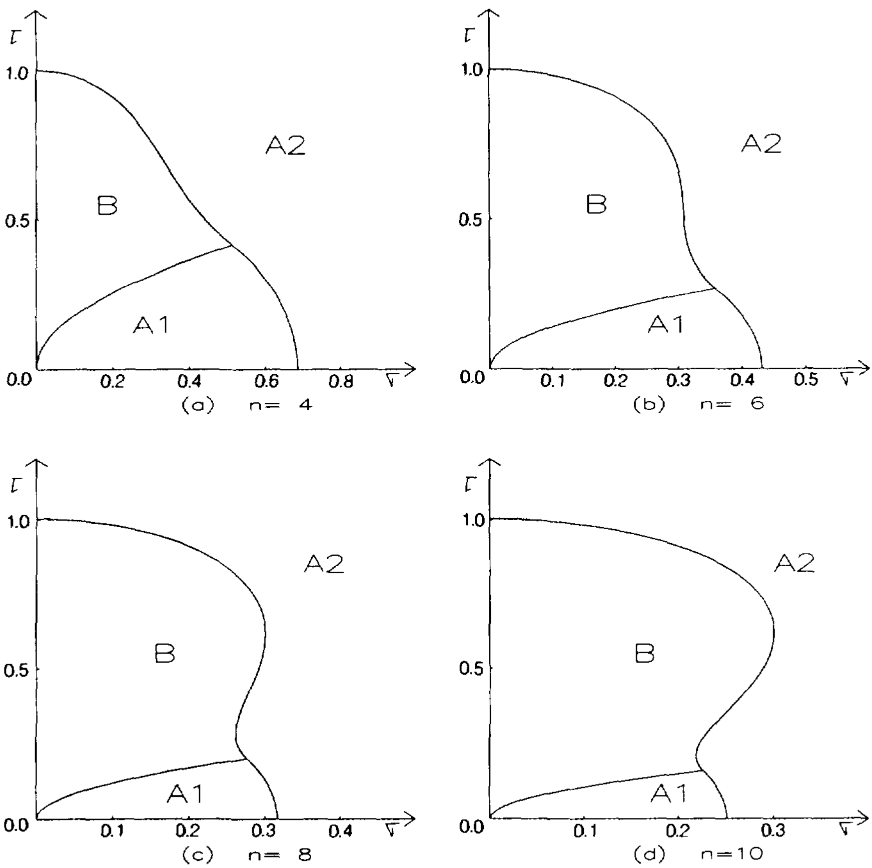

FIG. 1. Domains A_1, A_2, and B. (a) $n=4$; (b) $n=6$; (c) $n=8$; (d) $n=10$.

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